

A Small World Network of Prime Numbers

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According to Goldbach conjecture, any even number can be broken up as the sum of two prime numbers : $n = p + q$. We construct a network where each node is a prime number and corresponding to every even number n , we put a link between the component primes p and q . In most cases, an even number can be broken up in many ways, and then we chose *one* decomposition with a probability $|p - q|^\alpha$. Through computation of average shortest distance and clustering coefficient, we conclude that for $\alpha > -1.8$ the network is of small world type and for $\alpha < -1.8$ it is of regular type. We also present a theoretical justification for such behaviour.

I. INTRODUCTION

The study of networks, with an emphasis on small-world behaviour and scale invariant properties has turned out to be very important for analysing the statistical properties of diverse type of systems [1]. A network is defined as a graph consisting of some “nodes” and some “links” (or edges). When each pair of node is connected by a link, the network becomes a trivial one. One therefore links (or does not link) two nodes according to some intrinsic property of them. Depending on the context, different properties of the nodes are relevant in different networks, leading to different rules for linking. For example, in science collaboration network, each scientist is a node and two nodes (scientists) are linked when they are co-authors in at least one paper. In English language network, each word is a node and a pair of words are linked if (in one or more sentences) they appear side by side or one word apart. In this way, a network structure can be identified in widely varied contexts. Once a network is identified, one can measure in it some characteristic properties like the average shortest distance, clustering coefficient, degree distribution etc. When the degree distribution decays as a power law, the network is said to be scale-free. When the average shortest distance is small (increases only logarithmically with the size of the network) but the clustering coefficient is high compared to the random network, the network is called small world. Many natural and man-made networks [1] have been found to be scale-free and/or small-world.

Recently, Corso [2] has considered a network where each node is a natural number and two nodes are linked if they share a common prime factor larger than some chosen lower limit. The network is not scale-free (except in some restricted sense) but is a small-world unless the lower limit is 1, that is, unless one considers all prime factors. Motivated by this study, we consider here a network where each node is a prime number. The rule of placing links will be explained in the next section. The rule relies upon the validity of what is known as Goldbach conjecture [3] and involves a tunable parameter. Depending on the value of the parameter, we have a small-world or a regular network. We mention that our work has

no connection with the issue of the *validity* of Goldbach conjecture.

In the next section, we shall describe the network and present a theoretical analysis of its behaviour. In Section III we shall describe the computational studies and in Section IV present the general conclusions.

II. THE MODEL

Goldbach conjecture says that any even number (> 2) can be written as the sum of two prime numbers (often in more than one way). To construct our network, we start with the even number 8 and note that it can be broken up into primes as $8 = 3 + 5$. (For avoiding uninteresting complications we do not consider even numbers below 8.) We put (the first) two nodes in the network and label them as 3 and 5 and put a link between them. Now we consider all even numbers 10, 12, 14, \dots upto some N_e and break up each of them into primes as $n = p + q$. If p is not already in the network, a new node labelled as p is added, and similarly for q . Then a link is put between p and q . A complication is that (as mentioned earlier) very often one even number can be broken up into primes in more than one way. If one puts links corresponding to all the decompositions, then one has a link between almost every pair of nodes and the network becomes trivial. Therefore, with the following prescription we choose *one* way of decomposition for every even number, depending on the *difference* between the component primes. We calculate the difference $\Delta = |p - q|$ between the two components for every break-up and choose one break-up with probability Δ^α , where α is a (in fact, the only) parameter of the model. For example, the number $n = 24$ can be broken up in three ways : $5+19$, $7+17$, and $11+13$, with $\Delta=14$, 10 , and 2 respectively. In a large number of realisations, the link for 24 will be between 5 and 19 with probability p_1 , between 7 and 17 with probability p_2 , and between 11 and 13 with probability p_3 , where $p_1 = 14^\alpha/s$, $p_2 = 10^\alpha/s$ and $p_3 = 2^\alpha/s$ with $s = 14^\alpha + 10^\alpha + 2^\alpha$. As $p_1 + p_2 + p_3 = 1$, we could realise the choice of prime-pair by calling a random number between 0 and 1.

One should note that since we put one link for each even number, for M even numbers one will have exactly

M links, but some N ($< M$) number of nodes. One should also note that through the parameter α we actually control the difference between the prime pairs (chosen to be linked) for each even number. Thus, for $\alpha = 0$, our choice is independent of the difference Δ , while for $\alpha = -\infty$ ($+\infty$) the break-up with the smallest (largest) difference between the components is chosen.

What type of network we have thus constructed? To have an answer analytically, let us calculate the average value of Δ for a given even number n . This is

$$\langle \Delta \rangle = \frac{1}{\Omega} \sum \Delta^{\alpha+1},$$

where the sum extends over all Goldbach pairs of n and Ω is the number of such pairs. For positive values of the power $(\alpha + 1)$, the value of this sum will be dominated by the terms with large Δ and $\langle \Delta \rangle$ will hence increase as n increases. One of the chosen Goldbach pair will then become small, and these small numbers will be highly populated nodes. The network may therefore be expected to be of small-world type. On the other hand, for negative values of $(\alpha + 1)$, the value of the sum will be dominated by small Δ values and for large n the quantity $\langle \Delta \rangle$ will converge to some finite number. Both the chosen Goldbach pairs will be large (so that the difference between them remains small) and no node will be very highly populated. The network is then likely to lose the small-world character. Although Δ runs over some (but not all) of the natural numbers, the convergence behaviour of $\langle \Delta \rangle$ may be expected to be the same as the Riemann zeta function $\zeta(-\alpha - 1)$. As this function converges only for $\alpha < -2$, the change-over in the behaviour of the network is expected to occur around $\alpha = -2$.

III. SIMULATION STUDIES

To analyse the properties of the network by computer simulation, we measure several characteristics of the network. (i) *Average shortest distance between two nodes (d)* : The shortest distance between two nodes is the smallest number of links via which one can go from one node to the other. We have measured this quantity for all pairs of nodes and taken the average. Results for the measurement of this quantity is presented in Fig. 1 as a function of the number of nodes (N). It is observed that this quantity varies linearly with logarithm of the number of nodes as it happens for a small-world network. This behaviour prevails for all values of α upto a lower limit of $\alpha_0 = -1.8$. In particular, as α varies from 5 to 1, the d - N (log-linear) plot moves upwards parallel to itself. As α decreases further, the lines continue to move upwards but the slope increases continuously until for $\alpha < \alpha_0$ the line ceases to be straight and starts bending upwards. In this region, d varies linearly with N , as it happens for a regular network.

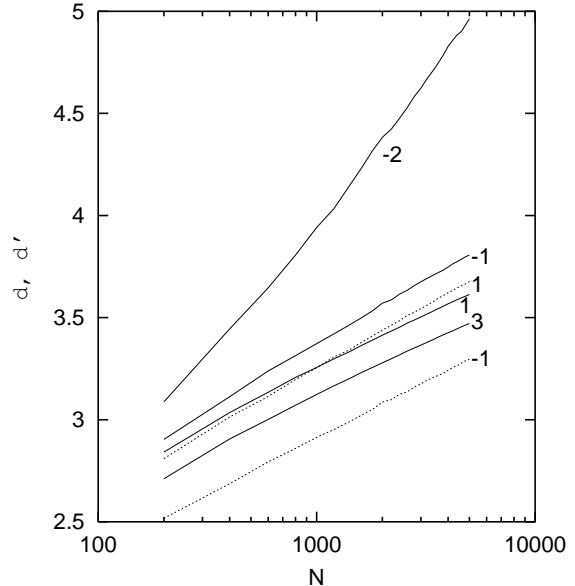


FIG. 1. Average shortest distance (d) as a function of the number of nodes for the network constructed from prime numbers (continuous line). Also shown is the same plot for a random network having the same number of nodes (d' , dotted line). The lines for d and d' cross over at $\alpha = 1$ and $N = 1000$. The numbers labelling the curves stand for the value of α . All results presented in this paper have been averaged over about 20 realisations of the network.

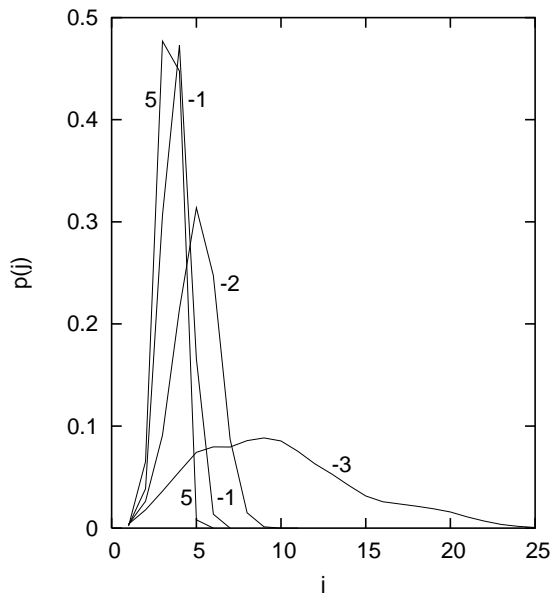


FIG. 2. Plot of $p(j)$ as a function of j , where $p(j)$ is the probability that a pair of nodes chosen randomly from the network will be j distance apart. The numbers labelling the curves stand for the value of α . $N = 5000$.

We have also plotted in Fig. 1, the average shortest distance (d') for a random network having the same number of nodes. As is well-known [1], the $d' - N$ (log-linear) plot is a straight line for the entire range of values of α and the lines maintain a constant slope and move gradually upwards as α increases. (However, for $\alpha > 2$ the $d' - N$ plot does not depend much on the value of α .)

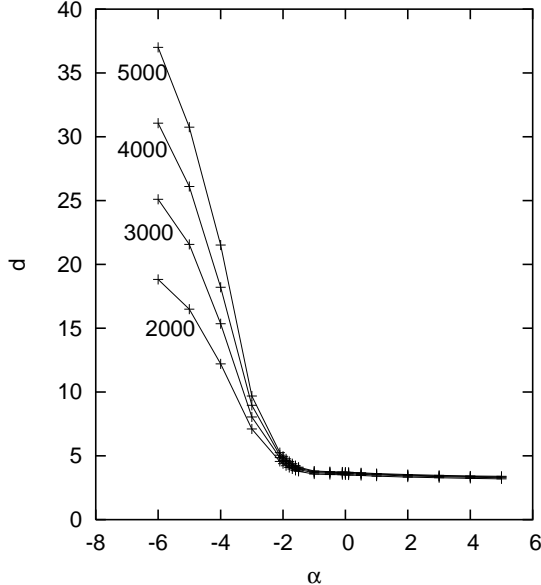


FIG. 3. Plot of average shortest distance d as a function of α for different values of the number of nodes N . The numbers labelling the curves stand for the value of N .

To gain further insight into the change of behaviour of d as a function of α , we have measured (Fig. 2) the probability $p(j)$ that a pair of nodes chosen randomly from the network will be j distance apart. It is observed that for $\alpha > \alpha_0$, $p(j)$ is high only for small values of j , indicating that most of the pairs of nodes are at a small distance apart and d (which is nothing but $\sum_j j p(j)$) is small. On the other hand, for $\alpha < \alpha_0$, the distribution $p(j)$ is flat over a large range of values of j , indicating that the distance between a pair of nodes will also often be large and d will be high due to the contribution from high j -values.

Lastly, in order to ascertain how sharply the change of behaviour occurs at $\alpha = \alpha_0$, we plot d against α for different values of N (Fig. 3). The rise of d for $\alpha < \alpha_0$ becomes sharper and sharper as N increases. Our estimate of $\alpha_0 = -1.8$ is based on simulations of networks with at most 5000 nodes.

(ii) *Clustering coefficient (C)* : The clustering coefficient C_i for the node i , is defined as the ratio M_i/m_i where M_i is the actual number of links among the neighbours of the node i , and m_i is the number of all possible links among the neighbours of i . (Thus, if i has degree k_i , then $m_i = k_i(k_i + 1)/2$.) The clustering coefficient

for the entire network is defined as the average of C_i over all nodes i . This quantity C has been measured in our network and compared with the same (C') for a random network with the same number of nodes (Fig. 4). It is found that $C > C'$ for the entire range of values of α investigated here. This behaviour, combined with the results for d , leads us to conclude that the network is of *small-world type* for $\alpha > \alpha_0$ and of *regular type* for $\alpha < \alpha_0$.

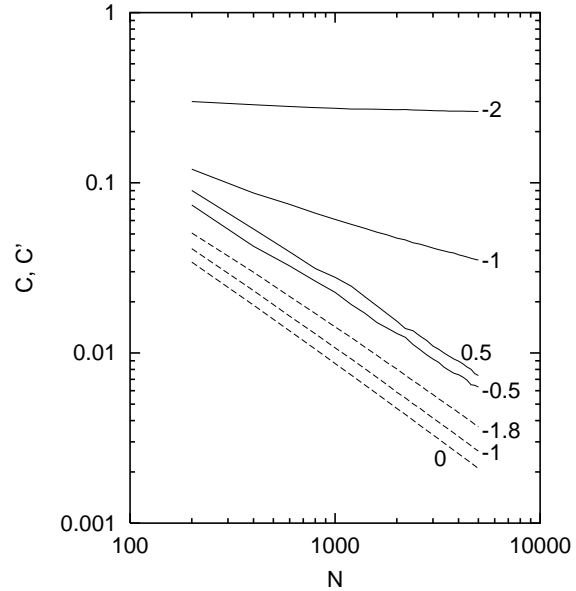


FIG. 4. Clustering coefficient (C) as a function of the number of nodes (N) for the network constructed from prime numbers (continuous line). Also shown is the clustering coefficient C' (dotted line) for a random network having the same number of nodes. For the entire range of α values C remains larger than C' . The numbers labelling the curves stand for the value of α .

The details of the behaviour of the clustering coefficient as a function of the number of nodes is as follows. For a given α , C decays algebraically with N and the $C - N$ line rises, maintaining a constant slope as α is increased from -0.5. But as α is decreased from -0.5, the line remains straight, rises upwards but becomes more and more horizontal. The slope almost vanishes (particularly in log-log scale) for $\alpha < \alpha_0$. For the corresponding random network, the clustering coefficient C' also decreases algebraically with N , and the $C' - N$ line rises, maintaining a constant slope, as α is decreased from 0 to α_0 . The lines for $\alpha < \alpha_0$ are almost coincident with those of $\alpha = \alpha_0$ and the lines for $\alpha > 0$ are also almost coincident with those of $\alpha = 0$.

(iii) *The degree distribution function $P(k)$* (defined as the probability that a node has k links attached to it) is of irregular type (Fig. 5) for all reasonable values of α , indicating that the network is not of scale-free nature.

However, some other related plots do contain some interesting information. Thus, it is interesting to observe how the network grows for different values of α (Fig. 6). For large positive values of α , by breaking up an even number n one chooses only those pairs of primes that are far apart. One of the chosen primes will therefore be a small prime number, while the other will be one that is close to n . Very often one will find that the latter prime has not been included in the network till now and thus a new node is added. The network then grows in size. On the other hand, for small (large negative) values of α , the difference between the primes chosen will be small. Each prime will then be $\sim (n/2)$ and will very often be found to be already present in the network. The network will then grow slowly. Such behaviour has been confirmed by simulation (Fig. 6).

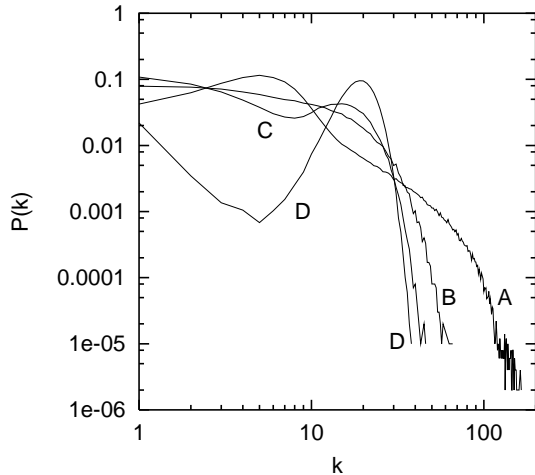


FIG. 5. Degree distribution $P(k)$ in the network constructed from prime numbers. The different lines correspond to $\alpha = 2$ (A), -0.1 (B), -0.5 (C), -2 (D). $N = 5000$.

We have also studied the α dependence (Fig. 7) of average connectivity $\langle k \rangle$ (which is simply $\sum_k kP(k) = 2M/N$, M being number of links and N being number of nodes) and fluctuation in connectivity defined as

$$f(k) = \sqrt{\langle k^2 \rangle - \langle k \rangle^2}.$$

Both these quantities display a sharp change at $\alpha = \alpha_0$. For a given number of nodes, the value of $\langle k \rangle$ will be large for $\alpha < \alpha_0$ and small for $\alpha > \alpha_0$ since there are more links in the former case than in the latter. For a regular network, the nodes have the same degree rather uniformly, so that $f(k)$ is small, but for a small-world network, some nodes are very rich in degree while the other nodes have low degree, rendering $f(k)$ very large. Moreover, for small-world network ($\alpha > \alpha_0$) the rich nodes

go on gaining links as the network evolves, so that the degree of the most-connected node (k_m , say) increases with size of the network (Fig. 8). In the regular network regime, no node is preferentially linked and k_m does not increase very much with N and in fact approaches the average connectivity $\langle k \rangle$.

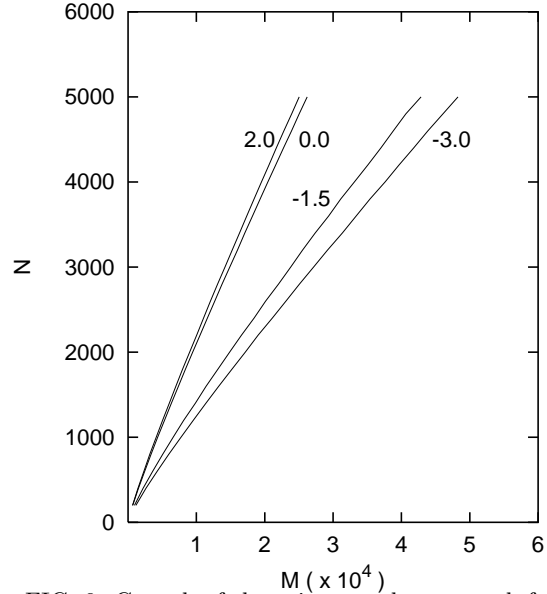


FIG. 6. Growth of the prime number network for different values of α . The number of nodes (N) is plotted as a function of the number of links (M). Since at each time step one link is added, the X-axis also represents the time step. The numbers labelling the curves stand for the value of α .

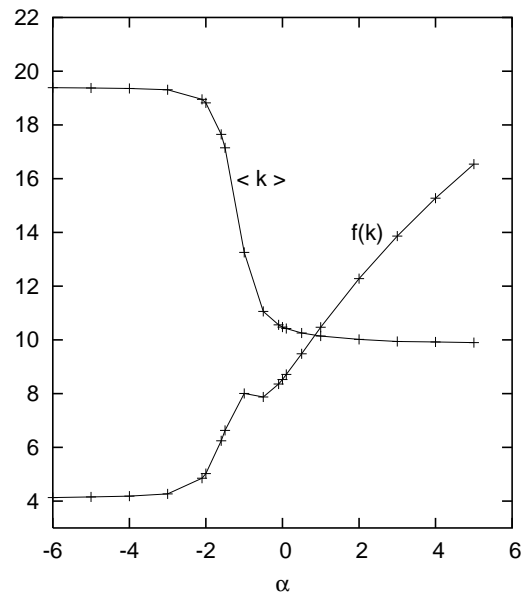


FIG. 7. Average connectivity $\langle k \rangle$ and fluctuation in connectivity $f(k)$ as a function of α . $N = 5000$.

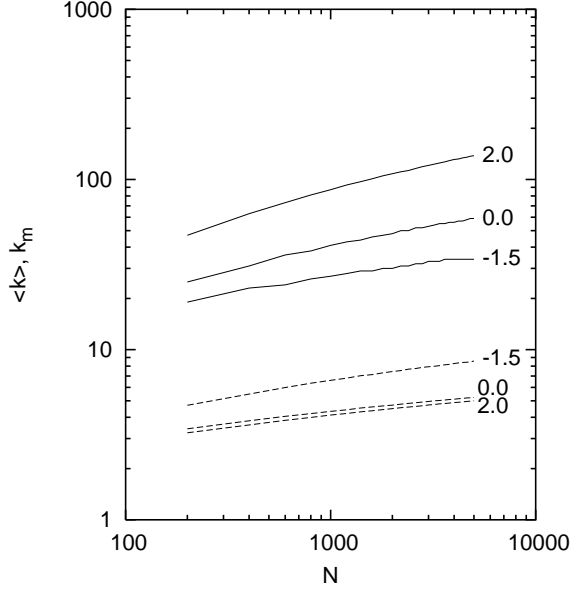


FIG. 8. Average connectivity $\langle k \rangle$ (broken line) and connectivity of the best connected node k_m (continuous line) as a function of the number of nodes (N) for different values of α . As α decreases, the $\langle k \rangle - N$ line rises, while the $k_m - N$ line descends. The numbers labelling the curves stand for the value of α .

(iv) *Clustering coefficient $C(k)$ measured as a function of the degree* of a node has been proved to be an important characteristic for many real networks [1,4]. This quantity is defined as C_i (as defined above) averaged by running i over only those nodes that have degree k . For $\alpha < 0$, this quantity has a peak at a small (< 20) value of k , indicating that the nodes mostly have low degree and high clustering coefficient. On the other hand, for $\alpha > 0$, $C(k)$ is almost flat extending over a large range of values of k . (Fig. 9)

(v) *Degree-degree correlation function r* , as proposed by Newman [5] measures the tendency of a link to have same type of degree (both high or both low) at its two ends. Thus, when r is positive, the network is assortative, and a link prefers to have the same type of node at the two ends whereas, when r is negative, the network is disassortative, and a link prefers to have different type of nodes (one of high degree and the other of low degree) at the two ends. This parameter may be measured from the relation [5]

$$r = \frac{M^{-1} \sum_i j_i k_i - [M^{-1} \sum_i (j_i + k_i)/2]^2}{M^{-1} \sum_i (j_i^2 + k_i^2)/2 - [M^{-1} \sum_i (j_i + k_i)/2]^2}$$

where j_i and k_i are the degrees of the nodes that are at the two ends of the i -th link. For the network under study, the quantity r has been found to be positive (negative) for negative (positive) values of α (Fig. 10). This indicates that the nature of the network changes from assortative to disassortative as α changes its sign.

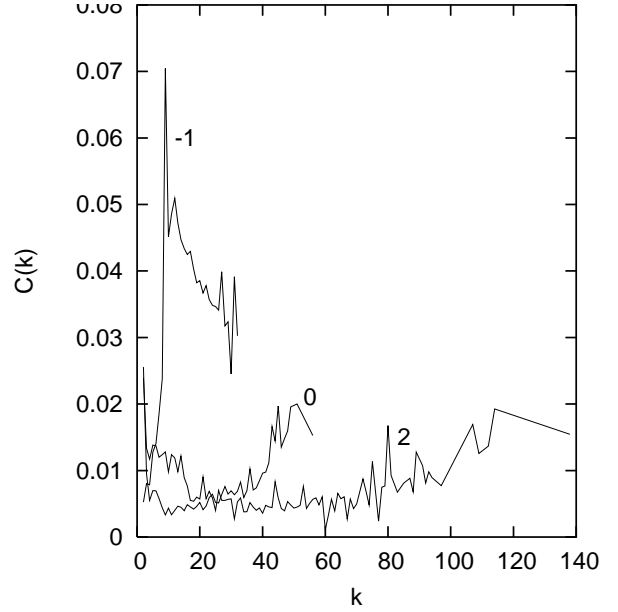


FIG. 9. Clustering coefficient $C(k)$ measured as a function of the degree. The numbers labelling the curves stand for the value of α . For $\alpha < -1$ the curve looks similar to that for $\alpha = -1$ but goes higher up and for $\alpha > 2$ the curve becomes more flat and moves further down. $N = 5000$.

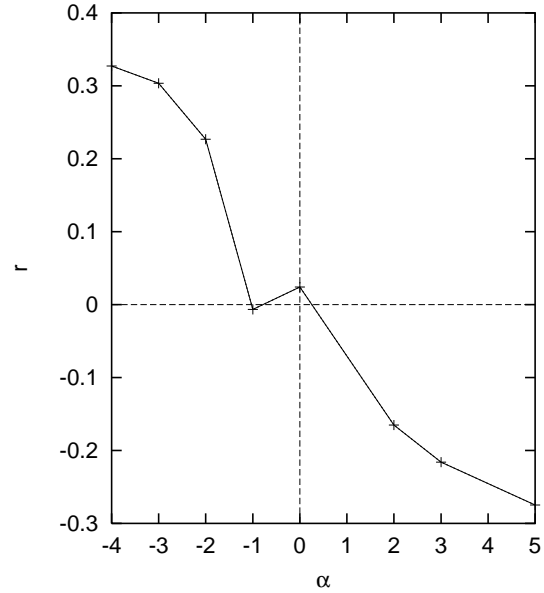


FIG. 10. Degree-degree correlation coefficient r as a function of α . Note that r bears a sign opposite to that of α . $N = 5000$.

IV. CONCLUSION

In conclusion, we have constructed a network of prime numbers with links placed on the basis of Goldbach con-

jecture. The network is of small world type when a parameter α of the model is larger than -1.8 and of regular type when α is lower than -1.8 . One must note that, for $\alpha > 0$ larger values of Δ (difference between the component primes) are preferred and the addition of a new link leads to a large prime getting attached to a small one. Thus, preferential attachment to small primes are supported and the ‘rich gets richer’ principle leads to a small-world network, as for the case of Barabasi-Albert network [1] although the scale-free property is not observed. In any case, the small-world property indicates some pattern in the Goldbach decomposition of prime numbers.

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